Scattering of Electromagnetic Waves by Turbulent, Weakly Ionized Plasmas *

W. Kröll

Institut für Theoretische Physik der Ruhr-Universität Bochum

(Z. Naturforsch. 23 a, 19—25 [1968]; received 9 July 1967)

The scattering of a plane, monochromatic electromagnetic wave by the fluctuations of the dielectric constant of a turbulent weakly, ionized plasma is investigated on the basis of the statistical theory of locally isotropic tubulence. The effects of collisions on the fluctuating dielectric constant of the electrons are taken into account. The spectral density determining the scattering cross section is related to the structure functions of the fluctuations of the electron concentration and the temperature. These structure functions are calculated from the transport equations with the use of methods developed in the statistical theory of strong turbulence.

The scattering of electromagnetic waves by the fluctuating dielectric constant of a plasma is of special importance in astrophysics, radioastronomy and in the diagnostics of laboratory plasmas. Scattering cross sections have been quantitatively calculated under two basic assumptions: 1) the plasma satisfies certain equilibrium conditions (meta-equilibrium); 2) the correlation of the fluctuations is due to the electrostatic interaction of the plasma particles and weak.

For turbulent plasmas the above conditions concerning the plasma state and the cause and strength of correlations are not valid. The scattering properties of such plasmas have been investigated by some more or less heuristic theories developed especially with respect to the scattering of radio waves by the lower layers of the ionosphere [1, 2]. But there are also attempts to treat the scattering of electromagnetic waves by turbulent plasmas on the basis of a statistical theory of turbulence [3]. These investigations consider the turbulent plasma as an incompressible fluid and calculate the scattering due to fluctuations of the electron density. They do not account for the effects of particle collisions on the fluctuating dielectric constant.

In this contribution we will use the statistical theory of locally isotropic turbulence developed by Kolmogorov, Oboukhov and Yaglom to determine the scattering cross section of a weakly ionzed, collision dominated plasma with temperature and density fluctuations. We consider strong turbulence in the sense of classical hydrodynamics.

System

Subject of this investigation is the scattering of a plane, monochromatic electromagnetic wave

$$\boldsymbol{E} = \boldsymbol{E}_0 \, e^{i(\omega t - \boldsymbol{k}\boldsymbol{r})} \tag{1}$$

by the randomly fluctuating scalar dielectric constant of a turbulent plasma within a volume V. The plasma is weakly ionized and collision dominated (seeded atmosphere). We assume that it can be described within the frame of the first order of the Chapman-Enskog development (hydrodynamic approximation). We neglect heat transfer by radiation and by electron diffusion as well as dissipation of mechanical energy by viscous forces. The turbulence is considered to be homogeneous and locally isotropic.

Concept

With the use of Maxwell's equations the scattering cross section is expressed by the spectral density of the fluctuating dielectric constant. Accounting for the effects of electron-neutral and electron-ion collisions we relate the fluctuations of the electron dielectric constant to the fluctuations of the temperature and the electron concentration. The spectral densities of these fluctuations are calculated from the transport equations using the statistical theory of locally isotropic turbulence.

* Most of this work was performed during a stay at the Department of Aerospace Engineering Sciences of the University of Colorado.

1 H. G. Booker and W. E. Gordon, Proc. Inst. Engrs. **38**, 401 [1950].

2 F. Villars and V. F. Weisskopf, Phys. Rev. **94**, 232 [1954].

3 R. A. Silverman, J. Appl. Phys. **27**, 699 [1956].

Analysis

Scattering Cross Section

The cross section $\sigma \, d\Omega$ for the scattering of a plane, monochromatic wave per unit plasma volume into the solid angle $d\Omega$ around the spatial direction $\boldsymbol{k}'$ is defined by

$$\sigma \, d\Omega = \frac{R^2}{V} \overline{|\boldsymbol{E}'|^2} / |\boldsymbol{E}_0|^2 \, d\Omega \,. \tag{2}$$

R is the distance from the scattering volume to the observation point, which is supposed to be large in comparison to the dimensions of the scattering region ($R \gg V^{1/3}$). $\boldsymbol{E}'$ is the electric field in the wave scattered in the direction $\boldsymbol{k}'$. The prime indicates quantities referring to the scattered wave. The bar means averaging over all realizations of the corresponding quantity.

We assume that $\boldsymbol{E}'$ and the electric induction $\boldsymbol{D}'$ are small in comparison to the corresponding fields in the incident wave and that the fluctuating part ε_1 of the dielectric constant is small in comparison to its average value $\bar{\varepsilon}$.

$\boldsymbol{E}'$ can then be calculated from Maxwell's equations via Fourier analysis in time and space [4].

$$\boldsymbol{E}' = \frac{-\exp i(\omega t - \boldsymbol{k}'\boldsymbol{R})}{R} \cdot \boldsymbol{k}' \times \boldsymbol{k}' \times \int \varepsilon_1 \boldsymbol{E}_0 \exp(i \boldsymbol{K} \boldsymbol{r}) \, d\boldsymbol{r} \tag{3}$$

with $\boldsymbol{K} = \boldsymbol{k}' - \boldsymbol{k}$. The integral in equ. (3) is to be extended over the whole scattering volume.

If we introduce the angle χ between $\boldsymbol{E}_0$ and $\boldsymbol{k}'$ and assume that the scattering causes only a small change in frequency ($k \approx k'$) equation (3) takes the form

$$\boldsymbol{E}' = \frac{-\exp i(\omega t - \boldsymbol{k}\boldsymbol{R})}{R} E_0 k^2 \sin\chi \int \varepsilon_1 \exp(i \boldsymbol{K} \boldsymbol{r}) \, d\boldsymbol{r} \,. \tag{4}$$

Inserting this expression into equation (2) we find

$$\sigma \, d\Omega = \frac{k^4}{V} \sin^2\chi \cdot \left(\int \overline{\varepsilon_1(\boldsymbol{r}_1) \, \varepsilon_1(\boldsymbol{r}_2)} \, e^{i\boldsymbol{K}(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \, d\boldsymbol{r}_1 \, d\boldsymbol{r}_2 \right) d\Omega \,. \tag{5}$$

$\overline{\varepsilon_1(\boldsymbol{r}_1) \, \varepsilon_1(\boldsymbol{r}_2)}$ is the correlation function C_ε of the refractive index fluctuations. Since our ε-field is homogeneous C_ε depends only on $\boldsymbol{r}_1 - \boldsymbol{r}_2$. Introducing the spectral density

$$S_\varepsilon(\boldsymbol{K}) = \int C_\varepsilon(\boldsymbol{r}_1 - \boldsymbol{r}_2) \, e^{i\boldsymbol{K}(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \, d(\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{6}$$

we finally obtain

$$\sigma \, d\Omega = k^4 \sin^2\chi \, S_\varepsilon(\boldsymbol{K}) \, d\Omega \tag{7}$$

and the evaluation of the scattering cross section is reduced to the problem of determining the spectral density of the fluctuations of the dielectric constant.

Dielectric Constant

We consider here the scattering due to fluctuations of the dielectric constant $\varepsilon^{(e)}$ of the free electrons. The contribution of the ions to ε may be neglected due to the large ion-electron mass ratio. The contribution of the neutrals becomes important only if the degree of ionization is extremely small.

The dielectric constant $\varepsilon^{(e)}$ is conveniently written in the form[5]

$$\varepsilon^{(e)} = 1 - g\left(\frac{\omega}{\bar{\nu}}\right) \frac{4\pi e^2 n_-}{m} \frac{1}{\omega^2 + \bar{\nu}^2} \tag{8}$$

where $g(\omega/\bar{\nu})$ is a slightly varying function with values close to 1 and the effective collision frequency $\bar{\nu}$ is defined by

$$\bar{\nu} = \bar{\nu}_0 + \bar{\nu}_i = \frac{2}{3\sqrt{2\pi}} \left(\frac{m}{KT}\right)^{5/2} \int_0^\infty [\nu_0(v) + \nu_i(v)] \, v^4 e^{-mv^2/2KT} \, dv \,. \tag{9}$$

$\bar{\nu}_0$ and $\bar{\nu}_i$ designate the effective frequencies of electron-neutral and electron-ion collisions resp. Neglecting the influence of the Ramsauer effect on ν_0 and evaluating ν_i with Rutherford's formula for the electron-ion collision cross section one obtains[5]

$$\bar{\nu}_0 = 8.3 \cdot 10^5 \pi \varrho^2 \sqrt{T} n_0 \,, \qquad \bar{\nu}_i = \frac{5{,}5}{T^{3/2}} n_+ \ln\left(\frac{220\,T}{n_+^{1/3}}\right) . \tag{10}$$

[4] L. D. LANDAU and E. M. LIFSHITZ, Electrodynamics of Continuous Media Pergamon Press, New York 1960.

[5] V. L. GINZBURG, Propagation of Electromagnetic Waves in Plasma, North-Holland Publ. 1961.

Where $\pi \varrho^2$ is an effective cross section. We now combine equs. (8) and (10). Introducing the electron concentration $c = n_-/n$ and the ideal gas law we find with the assumptions of quasi-neutrality and weak ionization

$$\varepsilon^{(e)} = 1 - \frac{4\pi c p e^2}{m K T} \frac{1}{\omega^2 + \left\{ \frac{8{,}3 \cdot 10^5 \pi \varrho^2 p}{K T^{1/2}} + \frac{5{,}5 c p}{K T^{5/2}} \ln\left(\frac{220\, T}{n_-^{1/3}}\right) \right\}} . \tag{11}$$

Transport Equations

Using the assumptions that the turbulence is homogeneous and that the fluctuations in electron concentration, temperature and pressure are small and not correlated with each other we find from equ. (11)

$$S_\varepsilon = \left(\frac{\partial \varepsilon}{\partial c}\bigg|_{\bar{y}}\right)^2 S_c + \left(\frac{\partial \varepsilon}{\partial T}\bigg|_{\bar{y}}\right)^2 S_T + \left(\frac{\partial \varepsilon}{\partial p}\bigg|_{\bar{y}}\right)^2 S_p , \quad y = \{c; T; p\} . \tag{12}$$

To determine the spectral densities of c and T we start from the corresponding transport equations. By taking appropriate moments of the Boltzmann equation we find up to first order in the Enskog development the following balances for particle densities, total momentum and total internal energy[6]

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \boldsymbol{v}) + \nabla \cdot (n_\alpha \boldsymbol{U}_\alpha) = R_\alpha , \quad \varrho \frac{\mathrm{d} v_i}{\mathrm{d} t} = - \sum_j \frac{\partial}{\partial x_j} p_{ij} , \tag{13, 14}$$

$$\frac{3}{2} \varrho \frac{\mathrm{d}}{\mathrm{d} t}\left(\frac{n K T}{\varrho}\right) = \sum_j \sum_i p_{ij} \frac{\partial v_i}{\partial x_j} + \nabla \cdot (\varkappa \operatorname{grad} T) - \frac{5}{2} \nabla \cdot \left(\sum_\alpha n_\alpha \boldsymbol{U}_\alpha K T\right) \tag{15}$$

with

$$n_\alpha \boldsymbol{U}_\alpha = - n \chi \operatorname{grad} \ln T - \frac{m_\beta}{\varrho} n^2 D \operatorname{grad} c_\alpha + \frac{m_\beta n^2 D}{\varrho}\left(1 - \frac{n m_\beta}{\varrho}\right) \operatorname{grad} \ln p \tag{16}$$

and

$$p_{ij} = p \delta_{ij} - \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - \left(\mu_b - \frac{2}{3}\mu\right) \delta_{ij} \nabla \cdot \boldsymbol{v} , \quad p = n K T . \tag{17, 18}$$

The indices α and β denote the different particle components. We distinguish here only two components: electrons and neutrals. This is possible, since we assume that the plasmas is weakly ionized and we are not interested in specific effects of the ions. The ions may be treated on equal grounds with the neutrals. n is the total particle density and $c_\alpha = n_\alpha/n$ the particle concentration of component α. ϱ is the mass density of the plasma and $\boldsymbol{v}$ the center of mass velocity. R_α denotes the particle production rate, $\varkappa$ the heat conductivity. D, χ, μ, μ_b are the coefficients of diffusion, thermal diffusion, viscosity and bulk viscosity resp.

Fluctuations of the Electron Concentration

Using our assumptions of weak ionization and small variations in temperature and density, equs. (13)—(18) may be considerably simplified. For the electron component the last term of equ. (16) vanishes due to weak ionization. Since the thermal diffusion coefficient χ is proportional to the electron concentration c_- the first term on the right-hand side of equ. (16) is small of second order and may therefore also be neglected. Consequently the particle balance for the electrons simplifies to

$$\frac{\partial n_-}{\partial t} + \nabla \cdot (n_- \boldsymbol{v}) - \nabla \cdot (n D \operatorname{grad} c_-) = R_- . \tag{19}$$

Multiplying equ. (19) with m_- and using the continuity equation

$$\frac{\partial \varrho}{\partial t} + \nabla (\varrho \boldsymbol{v}) = 0 \tag{20}$$

we find for the mass concentration γ_- of the electrons

$$\varrho \frac{\partial \gamma_-}{\partial t} + \varrho \boldsymbol{v} \operatorname{grad} \gamma_- - m_- n D \Delta c_- = m_- R_- . \tag{21}$$

In writing down the diffusion term of equ. (21) we have again made use of our assumption that the relative gradients of density and temperature are small. With the relations

$$m_- c_-/m_0 \approx \gamma_- , \quad \varrho \approx m_0 n \tag{22}$$

[6] J. W. Bond, K. M. Watson, and J. A. Welch, Atomic Theory of Gas Dynamics, Addison Wesley, 1965.

which hold due to weak ionization, we finally have

$$\partial c_-/\partial t + \boldsymbol{v}\,\mathrm{grad}\,c_- - D\Delta c_- = R_-/n\,. \qquad (23)$$

Starting from equ. (23) we can use a procedure developed by YAGLOM[7] to derive the structure function of the fluctuations of the electron-concentration. To do this we have to assume that the velocity field is approximately solenoidal, so that we can use the relation

$$\nabla \boldsymbol{v} = 0\,. \qquad (24)$$

Then we multiply equ. (23) with $c' = c_-(\boldsymbol{r}_2)$ and add the corresponding equation with the primed and unprimed coordinates interchanged. We obtain

$$\frac{\partial \overline{c' c}}{\partial t} = -2\sum_j \frac{\partial}{\partial \xi_j}\overline{v_j c' c} + 2\,D\,\Delta\overline{c' c} + 2\alpha\overline{c' c} \qquad (25)$$

where $\alpha_- = R_-/n_-$ is the number of ionizing collisions per electron and unit time, ξ_j are the components of the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and the Laplacian Δ has to be taken with respect to these components. We now introduce the correlation functions C_{cc}, C_{lc2} and the structure functions B_{cc}, B_{lcc} defined by

$$C_{cc} = \overline{c(\boldsymbol{r}_1)\,c(\boldsymbol{r}_2)}\,, \qquad C_{lc2} = \overline{v_l(\boldsymbol{r}_1)\,c^2(\boldsymbol{r}_2)}\,, \quad (26)$$
$$B_{cc} = \overline{(c(\boldsymbol{r}_1) - c(\boldsymbol{r}_2))^2}\,, \qquad B_{lcc} = \overline{(v_l - v_l')\,(c - c')^2}\,,$$

where v_l is the velocity component in the direction of $\boldsymbol{r}$. Due to assumption (24) C_{lc2} vanishes[8].

$$C_{lc2} = 0\,. \qquad (27)$$

We further have

$$\partial C_{cc}(r)/\partial t = \partial C_{cc}(0)/\partial t\,. \qquad (28)$$

This relation follows from the equality

$$B_{cc}(r) = 2\,(C_{cc}(0) - C_{cc}(r)) \qquad (29)$$

for a homogeneous and isotropic random field and the assumption that the structure function is time-independent.

From equ. (25) we obtain with the use of (27), (28) and (29)

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\,r^2\,B_{lcc}(r) - \frac{2D}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\,r^2\,\frac{\mathrm{d}B_{cc}(r)}{\mathrm{d}r} - 2\alpha\,B_{cc}(r) = 2\alpha\frac{\partial}{\partial t}C_{cc}(0) - 4\alpha\,C_{cc}(0)\,. \qquad (30)$$

This hierarchy equation is closed by the assumption of constant asymmetry a_c

$$B_{lcc} = -\,a_c\,B_{cc}\sqrt{B_{ll}}\,. \qquad (31)$$

B_{cc} can then be calculated provided that the longitudinal structure function B_{ll} of the velocity field is known.

B_{ll} is determined by the KOLMOGOROV equ.[9].

$$B_{lll} - 6\mu\frac{\mathrm{d}B_{ll}}{\mathrm{d}r} = \frac{6}{5}\,r\,\frac{\partial C_{ll}(0)}{\partial t} \qquad (32)$$

which follows from the Navier-Stokes equ. (14). With the assumption of constant asymmetry

$$B_{lll} = -\,a_l\,(B_{ll})^{3/2} \qquad (33)$$

one obtains from equ. (32) the relations

$$B_{ll} = \alpha_l r^2 = \frac{1}{10\,\mu}\frac{\partial C_{ll}(0)}{\partial t}\,r^2\,, \quad r \ll r_{01} = \left(\frac{\beta_l}{\alpha_l}\right)^{3/4}, \qquad (34)$$

$$B_{ll} = \beta_l r^{2/3} = \left(\frac{6}{5\,a_l}\frac{\partial C_{ll}(0)}{\partial t}\right)^{2/3} r^{2/3}\,, \quad r > r_{01}\,. \qquad (35)$$

Limiting r to small values such that equ. (34) holds and the conditions

$$4\,D/r^2 > a_c\sqrt{\alpha_l}\,, \quad O(a_c\sqrt{\alpha_l}) = O(\alpha) \qquad (36)$$

are satisfied, equ. (30) may be simplified to

$$\frac{\mathrm{d}^2 B}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}B}{\mathrm{d}r} + \left(\frac{3}{2}\,a_c\sqrt{\alpha_l} + \alpha\right)B = -\frac{A}{2D} \qquad (37)$$

where the constant A is given by

$$A = 2\,\partial C_{cc}(0)/\partial t - 4\,\alpha\,C_{cc}(0)\,. \qquad (38)$$

With the boundary condition

$$(\mathrm{d}B/\mathrm{d}r)_{r=0} = 0 \qquad (39)$$

[7] A. M. YAGLOM, Dokl. Akad. Nauk SSSR **69**, 743 [1949]. German translation in „Sammelband zur statistischen Theorie der Turbulenz“, Akademie-Verlag, Berlin 1958.

[8] A. M. OBUKHOV, Trudy Geofiz. Inst. Akad. Nauk SSSR **24**, 3 [1954]. German translation in „Sammelband zur statistischen Theorie der Turbulenz“, Akademie-Verlag, Berlin 1958.

[9] A. N. KOLMOGOROV, Ber. Akad. Wiss. UdSSR **32**, 1, 19 [1941]. German translation in „Sammelband zur statistischen Theorie der Turbulenz“, Akademie-Verlag, Berlin 1958.

we find the solution

$$B_{cc}(r) = -\frac{A}{12D} r^2, \quad r < \mathrm{Min}\left\{r_{01}, \left(\frac{4D}{a_c\sqrt{\alpha_l}}\right)^{1/2}\right\}. \tag{40}$$

In the range of large r values we combine equ. (35) with equ. (30) and transform to the new variable $x = r^{2/3}$. We then obtain

$$\frac{d^2B(x)}{dx^2} - \left(\frac{5}{2x} + \frac{3}{4}\frac{a\sqrt{\beta_l}}{D}x\right)\frac{dB(x)}{dx} + \left(\frac{21}{8}\frac{a_c\sqrt{\beta_l}}{D} + \frac{9\alpha}{4D}x\right)B(x) = -\frac{9}{8}\frac{A}{D}x. \tag{41}$$

If the conditions

$$D/x^2 \ll \tfrac{1}{2} a_c\sqrt{\beta_l}, \quad a_c\sqrt{\beta_l}/x \gg \alpha \tag{42}$$

are satisfied the solution of equ. (41) is

$$B = \frac{1}{\sqrt{x}}\exp\left\{-\frac{3}{16}\frac{a_c\sqrt{\beta_l}\,x^2}{D}\right\}\left\{C_1 M_{3/2,\,1/4}\left(\frac{3}{8}\frac{a_c\sqrt{\beta_l}\,x^2}{D}\right) + C_2 M_{3/2,\,-1/4}\left(\frac{3}{8}\frac{a_c\sqrt{\beta_l}\,x^2}{D}\right) - \frac{1}{3}\frac{A}{a_c\sqrt{\beta_l}}x\right\}, \tag{43}$$

where M designate Whittaker's functions. The constants C_1 and C_2 are to be determined by continuity requirements at the lower limit of the validity range of (43).

Due to the inequalities (42) and the asymptotic expansion

$$M_{3/2,\,\nu}(Z) = \frac{\Gamma(2\nu+1)}{\Gamma(\nu-1)} e^{Z/2} Z^{-3/2}\{1 + O(Z^{-1})\}, \quad Z \to \infty \tag{44}$$

the solution (43) is reduced to

$$B_{cc}(r) = -\frac{1}{3}\frac{A}{a_c\sqrt{\beta_l}} r^{2/3} = \beta_c r^{2/3}, \quad r > r_{0c} = \mathrm{Max}\left\{r_{01}; \left(\frac{3D}{a_c\sqrt{\beta_l}}\right)^{3/4}\right\}, \quad r < r_{1c} = \left(\frac{a_c\sqrt{\beta_l}}{\alpha}\right)^{3/2}. \tag{45}$$

For x values satisfying the conditions

$$D/x^2 \ll \tfrac{1}{3} a_c\sqrt{\beta_l}, \quad a_c\sqrt{\beta_l}/x \ll \alpha, \tag{46}$$

the solution of equ. (41) is given by

$$B_{cc}(x) = \exp\left\{-\frac{3\alpha}{a_c\sqrt{\beta_l}}x\right\}\eta(\xi) - \frac{A}{2\alpha} \tag{47}$$

where ξ is defined by

$$\xi = \sqrt{\frac{3}{4}\frac{a_c\sqrt{\beta_l}}{D}}\left(x - \frac{8\alpha D}{a_c^2\beta_l}\right) \tag{48}$$

and η is the solution of the equation

$$\eta'' + \xi\eta + \frac{12\alpha^2 D}{(a_c\sqrt{\beta_l})^3}\eta = 0. \tag{49}$$

Equ. (49) may be reduced to Whittaker's equation. Inserting the solution into (47) we see that in the range limited by (46) B_{cc} may be approximated by

$$B_{cc} = -A/2\alpha, \quad r > r_{1c} > r_{0c}. \tag{50}$$

As is easily verified the solutions (45) and (50) are asymptotic expansions of the function

$$B_{cc}(r) = -\frac{A}{2\alpha}\left\{1 - \frac{2^{2/3}}{\Gamma(1/3)}\left(\frac{r}{r_1}\right)^{1/3} K_{1/3}\left(\frac{r}{r_1}\right)\right\} \tag{51}$$

where K is the modified Bessel function of the second kind and the normalization length r_1 is defined by

$$r_1 = \left(\frac{3}{2}\right)^{3/2}\cdot r_{1c} = \left(\frac{3}{2}\frac{a_c\sqrt{\beta_l}}{\alpha}\right)^{3/2}. \tag{52}$$

We will therefore use equ. (51) as structure function of the turbulent fluctuations of the electron concentration in the whole range $r > r_{0c}$. The correlation is related to the structure function by equ. (29).

Temperature Fluctuations

We will now calculate the structure function of the temperature fluctuations. Neglecting the heat transfer due to particle diffusion and the heat production due to dissipation of mechanical energy by viscous forces we find from equ. (15) with the use of equs. (13) and (24)

$$\frac{dT}{dt} = -\frac{2}{3}T\,\nabla\cdot v + \frac{2}{K\,3n}\nabla(\varkappa\,\mathrm{grad}\,T) - \frac{T}{n}\sum_\alpha R_\alpha. \tag{53}$$

Within the frame of our model equ. (53) may be further simplified to

$$\frac{\partial T}{\partial t} + \boldsymbol{v}\,\mathrm{grad}\,T - \frac{2}{3}\frac{\varkappa'}{n}\Delta T = 0. \tag{54}$$

This equation has the same structure as equ. (23) without the production term. Therefore we use the same procedure as in the derivation of equ. (30). With the boundary conditions

$$B_{lTT}(0) = \frac{d}{dr}B_{TT}\,|_{r=0} = 0 \tag{55}$$

we obtain

$$B_{lTT} - \frac{4}{3}\frac{\varkappa'}{n}\frac{\mathrm{d}B_{TT}}{\mathrm{d}r} = -\frac{2}{3}r\frac{\partial C_{TT}(0)}{\partial t}. \quad (56)$$

This equ. too is closed by assuming constant asymmetry

$$B_{lTT} = -a_T B_{TT}\sqrt{B_{ll}}. \quad (57)$$

Inserting the longitudinal structure functions (34) resp. (35) we find

$$B_{TT}(r) = \alpha_T r^2 = \left(\frac{n}{4\varkappa'}\frac{\partial C_{TT}(0)}{\partial t}\right) r^2, \quad r < \operatorname{Min}\left\{r_{01}; \left(\frac{8}{3}\frac{\varkappa'}{n\sqrt{\alpha_l a_T}}\right)^{1/2}\right., \quad (58)$$

$$B_{TT}(r) = \beta_T r^{2/3} = \frac{2}{3 a_T \sqrt{\beta_l}}\frac{\partial C_{TT}(0)}{\partial t} r^{2/3}, \quad r > r_{0T} = \operatorname{Max}\left\{r_{01}; \left(\frac{8}{3}\frac{\varkappa'}{n\sqrt{\beta_l}}\right)^{3/4}\right\}. \quad (59)$$

Spectral Densites

Knowing the structure functions B_{cc} and B_{TT} we can now determine the spectral densities via the relation

$$B_{yy} = 2\int_{-\infty}^{\infty}(1-\cos(\boldsymbol{K}\boldsymbol{r}))\,S(\boldsymbol{K})\,\mathrm{d}\boldsymbol{K}. \quad (60)$$

This relation follows from equ. (29) with our assumption that the turbulence is homogeneous.

Since inhomogeneities with sizes much smaller than r_0 do not occur, $S(K)$ must rapidly go to zero for $K > 2/r_0$. The exact form of this decrease is uncertain, since the structure function is not known in the transition region $r \simeq r_0$. Therefore $S(K)$ is usually cut off at $2\pi/r_0$. The spectral densities corresponding to the structure functions (51) and (59) follow from equ. (60) to be

$$S_c(K) = -\frac{A}{2\alpha}\frac{\Gamma(8/3)}{4\pi^2}\frac{\sin(\pi/3)\, r_1^3}{(1+K^2 r_1^2)^{11/6}}, \quad (61)$$

$$S_T(K) = \frac{\Gamma(8/3)}{4\pi^2}\left(\sin\frac{\pi}{3}\right)\beta_T K^{-11/3}. \quad (62)$$

These relations are now inserted into equ. (12). The spectral density of the pressure fluctuations can be determined from the structure function (35) of the velocity field via the relation

$$\Delta p = -\varrho\sum_{i,j=1}^{3}\frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}. \quad (63)$$

But the effect of pressure fluctuations on the refractive index fluctuations is negligible in comparison to temperature and electron density fluctuations[10]. Thus we have from equ. (12)

$$S_\varepsilon = \frac{\Gamma(8/3)}{4\pi^2}\left(\sin\frac{\pi}{3}\right)\left\{-\left(\frac{\varepsilon-1}{c}\right)^2\left(\frac{2\bar{v}\bar{v}_i}{\omega^2+\bar{v}^2}-1\right)^2\frac{A}{2\alpha}\frac{r_1^3}{(1+K^2r_1^2)^{11/6}} + \left(\frac{\varepsilon-1}{T}\right)^2\left[\frac{\bar{v}(\bar{r}_0+3\bar{v}_i)}{\omega^2+\bar{v}^2}-1\right]^2\beta_T K^{-11/3}, \right. \quad (64)$$

$$R_0 = \operatorname{Max}\{r_{0c};\, r_{0T}\}, \quad K < 1/R_0$$

where all fluctuating quantities are to be represented by their average values. The lower K limit for the applicability range of (64) is determined by the validity of the assumptions of homogeneity and local isotropy.

The final result for the scattering cross section σ is obtained by inserting equ. (64) into equ. (7). Introducing the scattering angle via

$$K = 2k\sin\frac{\theta}{2} \quad (65)$$

we have for $1/R_0 > K > 1/r_1$

$$\sigma\,\mathrm{d}\Omega = 2.6\cdot 10^{-3}\left\{\left(\frac{\varepsilon-1}{c}\right)^2\left(\frac{2\bar{v}\bar{v}_i}{\omega^2+\bar{v}^2}-1\right)^2\beta_c + \left(\frac{\varepsilon-1}{T}\right)^2\left[\frac{\bar{v}(\bar{v}_0+3\bar{v}_i)}{\omega^2+\bar{v}^2}-1\right]^2\beta_T\right\} k^{1/3}\left(\sin\frac{\theta}{2}\right)^{-11/3}\sin^2\chi\,\mathrm{d}\Omega. \quad (66)$$

Discussion

On the basis of the statistical theory of locally isotropic turbulence we have rigorously calculated the scattering cross section of a weakly ionized, collision dominated turbulent plasma in the hydrodynamic regime. The hypothesis of constant asymmetry, which closes equs. (30), (32) and (56), has been supported by experimental findings. The absolute values a_l, a_c, a_T of the asymmetries enter-

[10] V. I. TATARSKI, Wave Propagation in a Turbulent Medium, McGraw-Hill, New York 1961.

ing the characteristics β_c, β_T of the fluctuating quantities c and T in equ. (66) remain undetermined within the frame of the theory used in this investigation. They have to be fixed either by experiments or by additional theoretical assumptions. The other quantities $\partial \overline{c^2}/\partial t$, $\partial \overline{T^2}/\partial t$ on which β_c and β_T depend, can be related to the mean square gradients[11]

$$\frac{\partial}{\partial t}\,\overline{T^2} = -\frac{4}{3}\,\frac{\pi'}{n}\,\overline{(\text{grad}\,T)^2}\,,\ \frac{\partial}{\partial t}\,\overline{c^2} = -2\,D\,\overline{(\text{grad}\,c)^2}. \tag{67}$$

$\overline{\partial v_l{}^2/\partial t}$ is by definition equal to $(2/3)\,\varepsilon$, where ε denotes the rate of energy dissipation per unit mass and unit time. The K and θ dependence of equ. (66) is the same as that found by SILVERMAN[3], who determined the scattering cross section by qualitatively applying the statistical theory of turbulence to the electron density. SILVERMAN did not account for the effect of collisions on the dielectric constant and neglected the influence of neutral gas density and temperature fluctuations.

Acknowledgments: I am grateful to Professor M. S. UBEROI for suggesting and promoting this work.

[11] A. M. OBUKHOV, Izv. Akad. Nauk SSSR, Ser. Geograf. Geofiz. **13**, 58 [1949]. German translation in „Sammelband zur statistischen Theorie der Turbulenz", Akademie-Verlag, Berlin 1958.

Dynamische Theorie der Röntgenstrahl-Interferenzen an schwach verzerrten Kristallgittern

II. Strahlenoptik von BLOCH-Wellen im allgemeinen Fall und im Zweistrahlfall

KYOZABURO KAMBE

Fritz-Haber-Institut der Max-Planck-Gesellschaft, Berlin-Dahlem *

(Z. Naturforsch. **23 a**, 25—43 [1968]; eingegangen am 5. September 1967)

A geometrical-optical approach to the problem of the propagation of X-rays in weakly deformed crystals is developed in a general form along the lines of the Hamilton-Jakobi theory. It starts from the eikonal equation, which has been derived from Maxwell's equations in Part I[1] and arrives at the equations of PENNING and POLDER[3] and Fermat's principle by KATO[4]. The amplitude equation, which has also been derived in Part I, is interpreted by means of the energy-flow picture, the absorption being taken into account.

Application of the theory to the two-beam case gives an equation of rays, which has a remarkable resemblance to the relativistic equation of motion of charged particles in an electromagnetic field. Representation of the lattice distortion by a displacement vector enables one to obtain equations of rays, phases and amplitudes in a form suited to the practical calculation.

Im ersten Teil[1] dieser Arbeit wurde gezeigt, daß sich die Fortpflanzung von Röntgen-Strahlen in schwach verzerrten Kristallgittern durch eine strahlenoptische Näherung beschreiben läßt. Durch den gleichzeitigen Übergang zu sehr hohen Frequenzen und sehr kleinen Gitterperioden wurden aus der Wellengleichung eine Eikonalgleichung und die zugehörige Amplitudengleichung abgeleitet.

Bei Kenntnis der Eikonalgleichung verfügt man über die üblichen mathematischen Methoden der geometrischen Optik[2] zur Berechnung von Strahlen, z. B. die Hamiltonschen kanonischen Gleichungen (PENNING und POLDER[3]) und das Fermatsche Prinzip (KATO[4]). Die *Strahlen* werden zunächst als Integrationswege für die Lösung der Eikonal- und Amplitudengleichung eingeführt. Sie haben darüber hinaus eine physikalische Bedeutung als Wege der Fortpflanzung schmaler Wellenbündel sowie der Energieströmung. Die Amplitudengleichung kann als Energieerhaltungsgesetz aufgefaßt werden.

Nachdem die Theorie im Teil A der vorliegenden Arbeit in allgemeiner Form entwickelt worden ist, wird sie im Teil B auf den Zweistrahlfall, nämlich den Interferenzfall an *einer* Netzebene, angewandt.

* Abteilung Prof. Dr. K. MOLIÈRE.

[1] K. KAMBE, Z. Naturforsch. **20a**, 770 [1965].

[2] M. BORN u. E. WOLF, Principles of Optics, Pergamon Press, London 1959. Kap. 3.

[3] P. PENNING u. D. POLDER, Philips Res. Rep. **16**, 419 [1961].

[4] N. KATO, J. Phys. Soc. Japan **18**, 1785 [1963]; **19**, 67, 971 [1964]. Das Fermatische Prinzip wurde von KATO aus den Maxwellschen Gleichungen mit Hilfe von „modifizierten Blochschen Funktionen" ebenfalls über eine Eikonalgleichung hergeleitet.